

# Correcting the effects of spontaneous emission on cold-trapped ions

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## Abstract

We propose two quantum error correction schemes which increase the maximum storage time for qubits in a system of cold trapped ions, using a minimal number of ancillary qubits. Both schemes consider only the errors introduced by the decoherence due to spontaneous emission from the upper levels of the ions. Continuous monitoring of the ion fluorescence is used in conjunction with selective coherent feedback to eliminate these errors immediately following spontaneous emission events, and the conditional time evolution between quantum jumps is removed by symmetrizing the quantum codewords.

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## I. INTRODUCTION

It was recognized over a decade ago by Deutsch [1] that a quantum computer has the potential to perform certain computational tasks much more efficiently than its classical counterparts, and the most striking example to date is the factorization of large numbers proposed by Shor [2]. The advent of quantum computation would also enable the realization of arbitrary quantum measurement processes and ultimately the efficient simulation of physical systems as envisaged by Feynman [3]. A measurement scheme recently presented by the authors [4] illustrates the usefulness of quantum computations in generating a “difficult” operator transformation in the context of an ion trap.

The possibility of building a quantum computer has received a lot of attention in the last few years, following several implementation proposals based on well-known physical systems. At this stage, ion traps [5] and optical cavities [6] are the leading candidates, and experimental work has progressed to the point of demonstrating the operation of two-bit quantum gates [7].

A major hurdle in the implementation of a quantum computer is that the presence of decoherence in any open physical system introduces random errors in the computation process, which will grow with time if left unchecked.

To illustrate the importance of this problem, Plenio and Knight [8] have considered the fundamental limitation imposed on Shor’s factorization algorithm by decoherence due to spontaneous emission. The maximum computation time  $T$  available on an  $N$ -qubit register is bounded by the decoherence time  $\tau_d$ , which can be expressed as

$$\tau_d = \frac{\tau_q}{N} \quad , \quad (1)$$

if the qubits are coupled to the environment independently, where  $\tau_q$  is the decoherence time of a single qubit. In the case of a linear rf trap, the switching rate of the laser pulses used to implement quantum gates is directly proportional to the Rabi frequency  $\Omega$ , which depends on the spontaneous emission rate  $\gamma$  of the qubit transitions ( $\Omega \propto \sqrt{\gamma}$ ), therefore

the execution time of an elementary logic gate ultimately depends on the qubit decoherence time  $\tau_q = \gamma^{-1}$ . Thus the dependence of the computation time on the number of input bits ( $\log_2 L$  bits for a decimal integer  $L$ ) was shown to be much stronger

$$T \propto (\log_2 L)^8 \quad , \quad (2)$$

than that obtained by assuming the logic gates can be performed in a time independent of the qubit decoherence time,

$$T \propto (\log_2 L)^3 \quad . \quad (3)$$

This result implies that factorizing a 4-bit number using Shor's algorithm would be very difficult to implement experimentally, whereas the factorization of nontrivial numbers with hundreds of bits appears impossible for the trapped-ion realization of a quantum computer.

Hence quantum error-correcting codes are necessary to run error-free computations, and a number of recent papers [9–12] have addressed this issue. The proposed codes allow for an arbitrary interaction between qubits and their environments, and prevent memory errors by redundantly encoding the information contained in logical qubits across an entanglement of several physical qubits. The resulting entangled states are known as **quantum codewords**, which can be decoded to give logical qubits, and the additional qubits required to implement the codes are labeled the **ancilla**.

The codewords can be recovered after each decoherence event using coherent feedback, given that only one qubit out of each codeword decoheres. Laflamme *et al.* [10] have shown that the most efficient perfect code which can correct all one-bit errors using a minimum number of ancillary qubits, consists of five-qubit codewords.

The problem of additional errors introduced by decoherence and inaccuracies during the error correction process was recently addressed by DiVincenzo and Shor [13]. The quantum gates used to implement error-correcting codes often depend on analog parameters, hence they need to be fault-tolerant in order to achieve robust computation which yields correct results even in the presence of moderate levels of noise.

In contrast to the general schemes referred to above, we only consider the errors arising from one source of decoherence, namely spontaneous emission, in a quantum computer based on an ion trap. This implies that our error-correcting codes are limited in scope, however they are directly applicable to experiment, and use a minimum of additional memory resources.

Our physical system consists of  $N$  ions confined in a linear rf trap, as proposed by Cirac and Zoller [5] for the implementation of a quantum computer. Each of the ions is laser-cooled into the Lamb-Dicke limit, and experiences harmonic motion at the trapping frequency  $\nu$  of a collective vibrational mode. Qubits are represented by two-level electronic transitions  $|0\rangle \leftrightarrow |1\rangle$  consisting of hyperfine levels in order to minimize the atomic recoil from spontaneous emission events. One-bit and two-bit quantum gates are realized by applying carrier and sideband Raman laser pulses to the appropriate ion(s), using the collective vibrational mode as a common bus to achieve ion-ion coupling.

The errors experienced by an  $N$ -ion register running a quantum computation can be conveniently divided into memory and gate errors. Memory errors affect qubits when they are simply stored in the trap, and are due to decoherence from a number of sources including spontaneous emission from qubits, vibrational heating due to technical imperfections in the trap, and collisions with residual gas molecules. Gate errors arise during quantum gate operations due to decoherence from the same sources as above, or from technical inaccuracies involved in applying a gate, such as the timing of laser pulses.

A quantum error correction scheme rendering quantum gates tolerant to errors introduced by vibrational damping of the ions was recently proposed by Cirac *et al.* [14]. Their scheme redundantly encodes logical qubits in four electronic levels of each ion, and inverts the effects of quantum jumps of the phonon number using projection measurements to provide non-unitary feedback and restore the initial state, so that the gate can be repeated. Fault-tolerance is achieved by applying a measurement projecting onto the ideal state, if no jump occurs during the gate operation. The size of the quantum network which can be realized without any errors is squared relative to the uncorrected case, since the first order effects of decoherence are eliminated.

In this article we present two quantum error-correcting codes which address only those memory errors introduced by spontaneous emission from the upper levels of qubits  $|1\rangle_j$  to the respective ground levels  $|0\rangle_j$ , at the decay rate  $\gamma_j$  for the  $j$ -th ion. Without any error correction, the maximum storage time for qubits in an  $N$ -ion register can be estimated by the decoherence time  $\tau_d = \gamma^{-1}/N$ , assuming that the spontaneous emission rates of the ions are equal but independent of each other. The upper levels are typically chosen to be metastable levels, hence the decoherence time for a single qubit ( $\tau_q = \gamma^{-1}$ ) can be as long as one minute.

In both of our schemes, the initial state of the ion register is prepared in an entangled state, ie. a quantum codeword  $|\psi\rangle_i$  which belongs to a logical Hilbert subspace  $\mathcal{H}_l$ . Quantum jumps corresponding to spontaneous emission events are continuously monitored using photodetectors positioned around the ion trap, and when a jump is detected, its effect on the system is immediately inverted using selective coherent feedback. Mabuchi and Zoller [15] have shown that the inversion of quantum jumps is possible if quantum codewords are uniquely mapped by the jumps into error states which can be transformed unitarily back to the logical subspace  $\mathcal{H}_l$ . We assume that the spontaneous emission events from different ions are distinguishable, so that the feedback process can be applied selectively to the system to recover the initial codeword  $|\psi\rangle_i$ , in a time much shorter than the decoherence time of the ion register, thus ensuring that no spontaneous emission occurs before the codeword is restored.

As emphasized by Plenio *et al.* [16] recently, the quantum codewords used for the register need to be invariant under the conditional time evolution between jumps, so that the state prior to each decay event is known to be a codeword. The conditional time evolution causes the ion register to evolve non-unitarily according to the transformation

$$U_c(t) = \exp\left(-\frac{\Gamma t}{2} \sum_j |1\rangle_j \langle 1|\right) , \quad (4)$$

where  $\Gamma$  is the spontaneous emission rate of all the excited levels, and the state of the register has been transformed to an interaction picture which eliminates the free evolution

of the ions. Thus the quantum codewords have to be restricted to a logical Hilbert subspace consisting of register states which have the same number of excited ions.

In the following section we outline the use of an alternative logical basis consisting of Fourier-transformed states, and present an error correction scheme for individual qubits based on entangling two ions for each codeword. We then consider an error correction scheme for an  $N$ -ion register, which uses complementary electronic number states to realize codewords in a very efficient manner.

## II. ERROR CORRECTION USING FOURIER-TRANSFORMED STATES

For the purposes of quantum computation with an ion trap, a Fourier-transformed basis  $\tilde{L}$  can be used as an alternative to the logical basis  $L$  comprising the electronic levels  $|0\rangle, |1\rangle$ . The Fourier-transformed states  $|\tilde{0}\rangle, |\tilde{1}\rangle$  making up  $\tilde{L}$ , are orthogonal superpositions of the electronic levels,

$$|\tilde{0}\rangle_i = \frac{1}{\sqrt{2}}(|0\rangle_i + |1\rangle_i) \quad (5)$$

$$|\tilde{1}\rangle_i = \frac{1}{\sqrt{2}}(|0\rangle_i - |1\rangle_i) \quad (6)$$

The basis  $\tilde{L}$  holds the advantage that both amplitude coefficients of an arbitrary qubit can be preserved when there is a spontaneous emission from the excited level  $|1\rangle$ , hence enabling us to recover the initial qubit.

One-bit quantum gates are implemented in basis  $L$  by applying a standing wave  $k$ -pulse  $V_i^k(\phi)$  (with laser phase  $\phi$ ) to the  $i$ -th ion,

$$V_i^k(\phi) = \exp \left[ \frac{-ik}{2} (|1\rangle_i \langle 0| e^{-i\phi} + |0\rangle_i \langle 1| e^{i\phi}) \right] \quad , \quad (7)$$

to rotate the respective electronic levels. The laser pulse is on resonance with the electronic transition  $|0\rangle_i \leftrightarrow |1\rangle_i$ , and the equilibrium position of the ion is placed at an antinode of the standing wave for the duration of the pulse. The same unitary transformation  $V_i^k(\phi)$  also implements rotations in  $\tilde{L}$ . For example, a laser pulse  $V_i^{\pi/2}(-\pi/2)$  can be used to swap between the basis states of  $L$  and  $\tilde{L}$ ,

$$|0\rangle_i \rightarrow |\tilde{0}\rangle_i \quad (8)$$

$$|1\rangle_i \rightarrow -|\tilde{1}\rangle_i \quad (9)$$

Cirac and Zoller [5] have shown how one-bit rotations and a series of sideband laser pulses, with the equilibrium positions of the ions at the nodes of the respective standing waves, can be used to implement a controlled-not gate  $U_{ij}$  in basis  $L$ , for two ions  $i, j$ ,

$$\begin{aligned} U_{ij}|0\rangle_i|0\rangle_j &= |0\rangle_i|0\rangle_j \\ U_{ij}|0\rangle_i|1\rangle_j &= |0\rangle_i|1\rangle_j \\ U_{ij}|1\rangle_i|0\rangle_j &= |1\rangle_i|1\rangle_j \\ U_{ij}|1\rangle_i|1\rangle_j &= |1\rangle_i|0\rangle_j \quad , \end{aligned} \quad (10)$$

employing a vibrational mode common to the ions. The control qubit  $i$  remains unchanged, whereas the target qubit  $j$  flips in the case that the control qubit is set to  $|1\rangle_i$ . An experimental realization of this quantum gate has been demonstrated recently by Monroe *et al.* [7]. The same transformation  $U_{ij}$  applied to the logical states of  $\tilde{L}$ , also implements a controlled-not gate,

$$\begin{aligned} U_{ij}|\tilde{0}\rangle_i|\tilde{0}\rangle_j &= |\tilde{0}\rangle_i|\tilde{0}\rangle_j \\ U_{ij}|\tilde{0}\rangle_i|\tilde{1}\rangle_j &= |\tilde{1}\rangle_i|\tilde{1}\rangle_j \\ U_{ij}|\tilde{1}\rangle_i|\tilde{0}\rangle_j &= |\tilde{1}\rangle_i|\tilde{0}\rangle_j \\ U_{ij}|\tilde{1}\rangle_i|\tilde{1}\rangle_j &= |\tilde{0}\rangle_i|\tilde{1}\rangle_j \quad , \end{aligned} \quad (11)$$

except that the control and target qubits are interchanged.

Since the necessary one-bit and two-bit quantum gates can be realized using known unitary transformations on the ion register, any arbitrary quantum computation can be implemented in the logical basis  $\tilde{L}$ . Given an arbitrary state  $|\psi\rangle_a = c_0|0\rangle_a + c_1|1\rangle_a$  for ion  $a$ , the corresponding qubit  $|\tilde{\psi}\rangle_a$  in the  $\tilde{L}$  basis,

$$|\tilde{\psi}\rangle_a = c_0|\tilde{0}\rangle_a + c_1|\tilde{1}\rangle_a \quad , \quad (12)$$

can be prepared by applying a  $V_a^{\pi/2}(-\pi/2)$  laser pulse, as shown in Eqs.(8-9).

The encoding process of our error correction scheme is implemented by entangling this arbitrary qubit with a second qubit from another ion  $b$ . Assuming that  $b$  is in the ground state  $|0\rangle_b$  initially, a  $V_b^{\pi/2}(-\pi/2)$  laser pulse is applied to prepare it in the logical state  $|\tilde{0}\rangle_b$ . Then a controlled-not gate  $U_{ba}$  is applied to the ions (using  $a$  as the control qubit) to obtain the desired codeword,

$$|\tilde{\psi}\rangle_i = c_0|\tilde{0}\rangle_a \otimes |\tilde{0}\rangle_b + c_1|\tilde{1}\rangle_a \otimes |\tilde{1}\rangle_b , \quad (13)$$

which can be maintained indefinitely using coherent feedback immediately following the detection of spontaneous emission events. When required, the initial qubit  $|\tilde{\psi}\rangle_a$  can be recovered from this codeword by applying another controlled-not gate  $U_{ba}$  to disentangle the ions.

If spontaneous emission occurs from the excited level of ion  $a$ , so that the state of the system is reduced to

$$c_0|0\rangle_a \otimes |\tilde{0}\rangle_b - c_1|0\rangle_a \otimes |\tilde{1}\rangle_b , \quad (14)$$

the amplitude coefficients of the original qubit are preserved, and the codeword  $|\tilde{\psi}\rangle_i$  can be restored by the coherent feedback process presented below, which is also illustrated in Figure 1.

First a  $V_a^{\pi/2}(\pi/2)$  laser pulse is applied to rotate the state of ion  $a$  into a logical state, i.e.,  $|0\rangle_a \rightarrow |\tilde{1}\rangle_a$ ,

$$c_0|\tilde{1}\rangle_a \otimes |\tilde{0}\rangle_b - c_1|\tilde{1}\rangle_a \otimes |\tilde{1}\rangle_b , \quad (15)$$

then a controlled-not gate  $U_{ab}$  is applied to entangle the ions,

$$c_0|\tilde{1}\rangle_a \otimes |\tilde{0}\rangle_b - c_1|\tilde{0}\rangle_a \otimes |\tilde{1}\rangle_b , \quad (16)$$

and finally a  $V_a^{\pi}(-\pi/2)$  laser pulse is applied to flip the logical states of qubit  $a$ ,

$$|\tilde{0}\rangle_a \rightarrow -|\tilde{1}\rangle_a \quad (17)$$

$$|\tilde{1}\rangle_a \rightarrow |\tilde{0}\rangle_a , \quad (18)$$



thus restoring the codeword  $|\tilde{\psi}\rangle_i$ .

We have assumed that we know which ion has emitted spontaneously, so that we can apply this feedback process selectively to the appropriate ions. If spontaneous emission occurs from the excited level of ion  $b$ , the same process is used to recover the codeword  $|\tilde{\psi}\rangle_i$ , with the ions  $a, b$  interchanged.

To achieve codeword invariance under the conditional time evolution given in Eq.(4), the number of excited states  $|1\rangle_j$  has to be constant for each elementary state in basis  $\mathcal{L}$  making up a codeword, so that the damping terms due to the conditional time evolution factor out without distorting the codeword. Thus the proposed codeword  $|\tilde{\psi}\rangle_i$  can be preserved between quantum jumps by a process of complementary extension, which results in a doubling of the register size.

The elementary codewords  $|\tilde{0}\rangle_a|\tilde{0}\rangle_b, |\tilde{1}\rangle_a|\tilde{1}\rangle_b$  can be made invariant with respect to the conditional time evolution by entangling each  $\mathcal{L}$ -basis product state of the ions  $a, b$  with its complementary product state from a second set of ions  $c, d$ , resulting in a symmetrization of the codewords. This encoding process can be achieved by setting the ions  $c, d$  in the excited state  $|1\rangle_c|1\rangle_d$  initially and applying two controlled-not gates  $U_{ac}U_{bd}$  to obtain the extended codeword  $|\Psi\rangle_i$ , which can be written in terms of the electronic states for basis  $\mathcal{L}$  as

$$\begin{aligned} |\Psi\rangle_i = & c_0 (|00\rangle_1|11\rangle_2 + |01\rangle_1|10\rangle_2 + |10\rangle_1|01\rangle_2 + |11\rangle_1|00\rangle_2) + \\ & + c_1 (|00\rangle_1|11\rangle_2 - |01\rangle_1|10\rangle_2 - |10\rangle_1|01\rangle_2 + |11\rangle_1|00\rangle_2) \quad , \end{aligned} \quad (19)$$

where  $|S_a S_b\rangle_1 |S_c S_d\rangle_2 = |S_a\rangle_a \otimes |S_b\rangle_b \otimes |S_c\rangle_c \otimes |S_d\rangle_d$ , and  $S_i = 0, 1$ .

However a new feedback process is now required to invert the effect of quantum jumps on this extended codeword. First a  $V^\pi(-\pi/2)$  laser pulse is applied to the ion from which the decay event is detected, and then the product states of all four ions are complemented to recover  $|\Psi\rangle_i$ , using the transformation

$$|S_a S_b\rangle_1 |S_c S_d\rangle_2 \rightarrow \frac{1}{\sqrt{2}} \left( |S_a S_b\rangle_1 |S_c S_d\rangle_2 + |\bar{S}_a \bar{S}_b\rangle_1 |\bar{S}_c \bar{S}_d\rangle_2 \right) \quad , \quad (20)$$

where  $\bar{S}_i$  is the complement of  $S_i$  (modulo 2). This feedback process has been adopted

in the following section to invert the effect of quantum jumps on codewords consisting of complementary electronic number states, in order to implement an improved error-correcting code, which requires less ancillary qubits.

### III. ERROR CORRECTION USING ELECTRONIC NUMBER STATES

The electronic number states  $|k\rangle_e$  of a system of  $N$  trapped ions form a logical basis, and are defined by the product of the individual electronic levels of the ions arranged in some definite order,

$$|k\rangle_e = |S_N\rangle_N \otimes |S_{N-1}\rangle_{N-1} \otimes \cdots \otimes |S_1\rangle_1, \quad (21)$$

where  $S_i = 0, 1$  represents a ground or excited state respectively. An  $N$ -ion register has  $2^N$  electronic number states labeled by the integer  $k = S_N \times 2^{N-1} + S_{N-1} \times 2^{N-2} + \cdots + S_1 \times 2^0$  ( $0 \leq k \leq 2^N - 1$ ) representing the binary string  $S_N S_{N-1} \dots S_1$ .

Spontaneous emission from the excited level  $|1\rangle_j$  of the  $j$ -th ion eliminates the amplitude coefficients of the number states  $|q\rangle_e$  which include the ground state  $|0\rangle_j$ , and are labeled by the integers  $q$  containing  $S_j = 0$ . However the amplitude coefficients of the complementary number states  $|\bar{q}\rangle_e$  are preserved, where  $\bar{q}$  is the complement of  $q$  (modulo  $2^N$ ). Therefore if the initial state  $|\psi\rangle_i$  is prepared so that the amplitude coefficients of the electronic number states  $|k\rangle_e$  and  $|\bar{k}\rangle_e$  are equal for all  $k$ ,

$$|\psi\rangle_i = \sum_{k=0}^{2^{N-1}-1} \frac{c_k}{\sqrt{2}} (|k\rangle_e + |\bar{k}\rangle_e), \quad (22)$$

then  $|\psi\rangle_i$  can be restored after a spontaneous emission event from any of the ions.

Given an arbitrary  $(N-1)$ -ion state

$$|\psi\rangle_i^{(N-1)} = \sum_{k=0}^{2^{N-1}-1} c_k |k\rangle_e^{(N-1)}, \quad (23)$$

the required  $N$ -ion codeword  $|\psi\rangle_i$  can be generated by coupling  $|\psi\rangle_i^{(N-1)}$  to the ground state of an additional ion

$$\sum_{k=0}^{2^{N-1}-1} c_k |0\rangle_N \otimes |k\rangle_e^{(N-1)} , \quad (24)$$

and complementing each N-ion electronic number state  $|k\rangle_e = |0\rangle_N \otimes |k\rangle_e^{(N-1)}$ . The original (N-1)-ion state  $|\psi\rangle_i^{(N-1)}$  can be recovered from the codeword by applying this transformation in reverse to disentangle the N-th ion from the other qubits. The complementing procedure is also used to restore the codeword after a spontaneous emission event, as shown in Figure 2, and its implementation is presented below.

We consider a three ion register to demonstrate our error-correcting code. An arbitrary 2-ion  $(a, b)$  state given by

$$c_0|0\rangle_e + c_1|1\rangle_e + c_2|2\rangle_e + c_3|3\rangle_e , \quad (25)$$

is transformed into the required codeword  $|\psi\rangle_i$  by coupling it to a third ion  $c$  and applying a complementing procedure,

$$|\psi\rangle_i = \frac{1}{\sqrt{2}}(c_0|0\rangle_e + c_1|1\rangle_e + c_2|2\rangle_e + c_3|3\rangle_e + c_3|4\rangle_e + c_2|5\rangle_e + c_1|6\rangle_e + c_0|7\rangle_e) , \quad (26)$$

where the electronic number states  $|k\rangle_e = |S_c\rangle_c \otimes |S_b\rangle_b \otimes |S_a\rangle_a$  for the ions  $a, b, c$ , are labeled by  $k = 4S_c + 2S_b + S_a$ .

Hence, if spontaneous emission occurs from the excited level of ion  $b$  for example, the codeword is reduced to

$$c_2|0\rangle_e + c_3|1\rangle_e + c_1|4\rangle_e + c_0|5\rangle_e , \quad (27)$$

and the original amplitude coefficients are preserved.

The feedback process necessary to restore the codeword  $|\psi\rangle_i$  consists of two steps. First a  $V_b^\pi(-\pi/2)$  laser pulse is applied to ion  $b$ , i.e.,  $|0\rangle_b \rightarrow |1\rangle_b$ , to match the amplitude coefficients with their initial electronic number states,

$$c_2|2\rangle_e + c_3|3\rangle_e + c_1|6\rangle_e + c_0|7\rangle_e , \quad (28)$$

and then each electronic number state is complemented, ie.  $|k\rangle_e \rightarrow \frac{1}{\sqrt{2}}(|k\rangle_e + |\bar{k}\rangle_e)$ , to obtain the codeword  $|\psi\rangle_i$ .

The required complementing transformation  $\mathcal{C}_N$  is an extension of the two-bit controlled-not gate to  $N+1$  ions, and therefore requires a vibrational mode common to all the ions. An extra ion  $x$  is prepared in the superposition  $\frac{1}{\sqrt{2}}(|0\rangle_x + |1\rangle_x)$ , and then entangled with the other ions. Each electronic number state  $|k\rangle_e$  coupled to ion  $x$ ,

$$\frac{1}{\sqrt{2}}(|0\rangle_x + |1\rangle_x) \otimes (|S_N\rangle_N \dots |S_1\rangle_1) , \quad (29)$$

is transformed by a series of controlled-not gates  $U_{x1}U_{x2}\dots U_{xN}$  (applied sequentially in any order), which leave qubit  $x$  unchanged,

$$|0\rangle_x |S_j\rangle_j \rightarrow |0\rangle_x |S_j\rangle_j \quad (30)$$

$$|1\rangle_x |S_j\rangle_j \rightarrow |1\rangle_x |\bar{S}_j\rangle_j , \quad (31)$$

while complementing  $|k\rangle_e$ , to transform the state from Eq.(29) into

$$\frac{1}{\sqrt{2}}(|0\rangle_x \otimes |S_N\rangle_N \dots |S_1\rangle_1 + |1\rangle_x \otimes |\bar{S}_N\rangle_N \dots |\bar{S}_1\rangle_1) , \quad (32)$$

where  $\bar{S}_j$  is the complement of  $S_j$  (modulo 2). We note that the controlled-not gates in the sequence  $U_{x1}U_{x2}\dots U_{xN}$  could be implemented simultaneously, using an  $(N+1)$ -bit quantum gate.

In the feedback process used to restore  $|\psi\rangle_i$ , a  $V_j^\pi(-\pi/2)$  laser pulse is first applied after spontaneous emission from the  $j$ -th ion, which sets the  $j$ -th digit of the remaining electronic number states to  $S_j = 1$ . Hence after applying the controlled-not sequence  $U_{x1}U_{x2}\dots U_{xN}$ , the ancillary ion  $x$  can be disentangled from the  $N$ -ion number states by applying one more controlled-not gate  $U_{jx}$ , using ion  $j$  which decayed spontaneously as the control qubit. The resultant state of the ion register is given by

$$|1\rangle_x \otimes \frac{1}{\sqrt{2}}(|S_N\rangle_N \dots |S_1\rangle_1 + |\bar{S}_N\rangle_N \dots |\bar{S}_1\rangle_1) , \quad (33)$$

for the electronic number state  $|k\rangle_N$ , thus realizing the required complementing transformation

$$\mathcal{C}_N |k\rangle_e = \frac{1}{\sqrt{2}}(|k\rangle_e + |\bar{k}\rangle_e) . \quad (34)$$

In the case that the complementing transformation is used to encode the arbitrary state  $|\psi\rangle_i^{(N-1)}$ , the  $N$ -th digit of the  $N$ -ion number states has been set to  $S_N = 0$  by preparing the  $N$ -th ion in the ground state, therefore the last step disentangling the ancillary ion should consist of applying the controlled-not gate  $U_{Nx}$ , using ion  $N$  as the control qubit.

Once again we have assumed that the continuous monitoring of spontaneous emission enables us to know which ion has emitted spontaneously, so that the coherent feedback process can be applied selectively. The resonant  $\pi$ -pulse  $V^\pi(-\pi/2)$  is applied to the particular ion from which spontaneous emission is detected, and is followed by the complementing procedure to recover the initial codeword.

As in the previous section, the codeword  $|\psi\rangle_i$  can be made invariant under the conditional time evolution between quantum jumps by entangling each electronic number state with its complementary number state from a second set of  $N$  ions. Thus the number of excited states  $|1\rangle_j$  is constant for each elementary  $2N$ -ion state making up the extended codeword, and equal to  $N$ . This encoding process can be achieved for  $|\psi\rangle_i$  by setting the second set of ions in the state  $|2^N - 1\rangle_2$  initially and applying a series of  $N$  controlled-not gates  $U_{a_1 a_2} U_{b_1 b_2} \dots$  to pairs of ions from the first (1) and second (2) sets, in order to obtain the symmetrized codeword  $|\Psi\rangle_{2N}$ ,

$$|\Psi\rangle_{2N} = \sum_{k=0}^{2^N-1} \frac{c_k}{\sqrt{2}} \left( |k\rangle_1 |\bar{k}\rangle_2 + |\bar{k}\rangle_1 |k\rangle_2 \right) . \quad (35)$$

In this case, the same feedback process presented above can be used to invert the effect of quantum jumps on the codeword  $|\Psi\rangle_{2N}$ , using the complementing transformation

$$\mathcal{C}_{2N} |k\rangle_1 |\bar{k}\rangle_2 = \frac{1}{\sqrt{2}} \left( |k\rangle_1 |\bar{k}\rangle_2 + |\bar{k}\rangle_1 |k\rangle_2 \right) , \quad (36)$$

which involves the electronic number states of all  $2N$  ions in the register.

#### IV. DISCUSSION AND CONCLUSION

The efficiency of the proposed error-correcting codes can be compared by considering their demands on memory and time resources. In both cases the requirement for ancillary

qubits is reduced compared to the general schemes discussed in Section I, which require the codeword encoding each logical qubit to use at least five qubits for “perfect” error correction, as shown by Laflamme *et al.* [10], and cannot correct for the conditional time evolution.

The quantum error correction scheme based on Fourier-transformed states can be extended to more logical qubits in a straightforward manner, eliminating the effects of spontaneous emission using four-qubit codewords, and thus storing  $N$  logical qubits on a  $(4N + 1)$ -ion register, taking into account the extra ion required for the complementing transformation.

On the other hand, the error-correcting code employing  $2N$ -ion complementary number states can store  $N$  logical qubits on a  $(2(N + 1) + 1)$ -ion register, and evidently uses available memory resources (trapped ions) more efficiently, especially for large numbers of logical qubits. However the price paid for this saving is that the feedback process involves acting on all the ions to correct a spontaneous emission event, rather than only four ions, hence this code requires more time (logic gate operations) and is harder to implement experimentally.

The time required for the feedback process can be estimated from the number of quantum gates used in its implementation, which is shown in Table I for the storage of  $N$  logical qubits, using both error correction schemes.

The controlled-not gate realized by Monroe *et al.* [7] in a Paul trap, has a duration of  $\tau_{cn} \approx 50\mu s$ , however it only operates on the qubits of a single ion and the associated vibrational mode. This accounts for approximately half of the laser pulses required to operate on two ions, since the state of the vibrational mode is not transferred to the second ion. The switching rate of the laser pulses implementing this logic gate is bounded by the characteristic vibrational frequency in the Paul trap ( $\nu \leq 100\text{MHz}$ ). We expect that in the first experimental realizations using a linear rf trap, the duration of a controlled-not gate between two ions will be greater than that obtained in a Paul trap, since the vibrational frequency will be of the order of only 10-100 kHz, however a value of  $\tau_{cn} \approx 100\mu s$  seems feasible eventually.

The time taken to realize a qubit rotation is inversely proportional to the Rabi frequency of the applied laser pulse. In the same article referred to above, Monroe *et al.* report that

the Rabi frequency of the carrier pulses was 140kHz, which results in interaction times of approximately  $10\mu\text{s}$  and  $20\mu\text{s}$ , for  $\pi/2$  and  $\pi$ -pulses respectively. The interaction times necessary to implement both qubit rotations and controlled-not gates, are expected to be reduced in future experiments as the laser power available is increased. However the maximum laser intensity is limited because at high intensities the two-level approximation and the rotating wave approximations will break down.

The feedback time for the error-correcting code using complementary electronic number states is proportional to the number of ions used, and is approximately given by  $(2N+1)\tau_{cn}$ . Therefore the time required to implement 5 logical qubits (using a 13-ion register) is  $\approx 1.1\text{s}$ , which is still short compared to the decoherence time  $\tau_d \approx 5\text{s}$  of the register (assuming a qubit decoherence time  $\tau_q \approx 1$  minute), and ensures that spontaneous emission is unlikely to occur before the codeword is restored.

In comparison, the error-correcting code using Fourier-transformed states only requires a constant time  $\approx 0.5\text{s}$  for its feedback process, though it requires a 21-ion register to implement 5 logical qubits.

There are a number of problems which can affect the performance of our schemes including inefficient detection of spontaneous emission events, gate errors during the feedback process and decoherence due to vibrational damping while the ions are stored. We have assumed that these problems will be manageable in future experiments by some combination of technical accuracy and error-correcting codes, in order to focus on how the errors due to a single fundamental source of decoherence can be corrected.

For example, the continuous monitoring of spontaneous emission over a  $4\pi$  solid angle cannot be realized efficiently with present technology since the photodetection efficiency is very small ( $< 10^{-2}$ ), and it is very difficult to cover the entire angle with detectors. The implementation of two-qubit logic gates in the feedback, encoding and decoding processes was also assumed to be perfect, but could be realized by making use of the fault-tolerant controlled-not gates proposed by Cirac, Pellizzari and Zoller [14].

In conclusion, we have proposed two quantum error correction schemes which rely on

continuous monitoring of spontaneous emission and selective coherent feedback to eliminate memory errors as soon as they are detected. A minimum number of ancillary qubits are required to realize codewords, by using a Fourier-transformed logical basis for the first scheme, and complementary electronic number states for the second scheme.

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# TABLES

error correction scheme using	number of qubit rotations	number of controlled-not gates
Fourier-transformed states	2	5
Complementary number states	2	$2N+1$

TABLE I. Number of elementary logic gates required to implement the error-correcting schemes for  $N$  logical qubits.

## FIGURES

FIG. 1. Flow chart of the encoding process for a codeword  $|\tilde{\psi}\rangle_i$ , and the required feedback process if a spontaneous decay event occurs from ion  $a$ .

FIG. 2. Flow chart of the encoding process for a codeword  $|\psi\rangle_i$ , and the required feedback process if the spontaneous decay event occurs from ion  $j$ , where the transformation  $\mathcal{A}_N$  represents the series of controlled-not gates  $U_{x1}U_{x2}\dots U_{xN}$ .

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